

# LARGE-SCALE SYSTEMS: COMPLEXITY, STABILITY, RELIABILITY\*

D. D. Šiljak\*\*

University of Santa Clara, Santa Clara, Calif. 95053

## INTRODUCTION

“Roughly, by a complex system I mean one made up of a large number of parts that interact in a nonsimple way” is Simon’s description (ref. 1) of a complex system. He goes on further to say that “complexity frequently takes the form of hierarchy,” and by intuitive arguments he points out that the evolution of complex systems is highly reliable if it is carried out as a hierarchic process whereby complex systems are formed by interconnecting stable simple parts (subsystems).

The main objective of this work is to show rigorously that a complex system with (or designed to have) a competitive structure has highly reliable stability properties. The competitive models were studied in such diverse fields as economics (refs. 2, 3) and biology (ref. 4), arms race (ref. 5), and pharmacokinetics (ref. 6), and only recently it was shown (refs. 7–10) that in these various scientific disciplines, such models are “fail-safe” stable. In the framework of the connective stability concept (refs. 11–14), we will provide a definite support of Simon’s intuitive arguments. We will show that, under certain conditions, a stable complex system can tolerate a wide range of nonlinear, time-varying perturbations.

For large-scale systems, a competitive hierarchic model will be constructed by aggregating the stability properties of each subsystem so that stability of the model implies stability of the original system despite structural perturbations whereby subsystems are disconnected and again connected in various ways during the operation of the system. That is, from stability of each subsystem and stability of the aggregate model on the upper hierarchic level, we infer connective stability of the overall complex system. This result is remarkable in that it provides a natural setting for designing large-scale dynamic systems as competitive structures with highly reliable stability properties.

The stability investigations of competitive models are carried out by using the powerful mathematical concept of the comparison principle and vector Liapunov functions (ref. 15). The concept is extended here to include considerations of the connective stability aspects and the effects of the structural perturbations. Furthermore, the decomposition-aggregation methods developed in this context can take advantage of the special structural features of complex systems and reduce considerably the dimensionality of relevant stability problems.

---

\*The research reported here was supported by NASA Ames Research Center, Moffett Field, Calif., under Grant NGR 05-017-010.

\*\*Department of Electrical Engineering and Computer Science.

## COMPETITIVE MODELS

Let us start with a linear constant dynamic system described by the differential equation\*

$$\dot{x} = \bar{A}x \quad (1)$$

where  $x(t) \in R^n$  is the state of the system and  $\bar{A} = (\bar{a}_{ij})$  is an  $n \times n$  constant matrix. By use of the connective stability concept (ref. 12), we derive the conditions under which stability is a highly reliable property of the system (1), and show that it remains stable despite a wide range of nonlinear, time-varying perturbations. In fact, by use of the modern mathematical machinery of the comparison principle (ref. 15), we show that, under relatively simple conditions, stability of the system (1) implies stability of a broad class of dynamic systems described by

$$\dot{x} = A(t, x)x. \quad (2)$$

In equation (2),  $x(t) \in R^n$  is again the state of the system and the  $n \times n$  matrix function  $A: T \times R^n \rightarrow R^{n^2}$  is defined, bounded, and continuous on  $T \times R^n$  so that the solutions  $x(t; t_0, x_0)$  of equation (2) exist for all initial conditions  $(t_0, x_0) \in T \times R^n$  and  $t \in T_0$ . The symbol  $T$  represents the time interval  $(\tau, +\infty)$ , where  $\tau$  is a number or the symbol  $-\infty$ , and  $T_0$  is the semiinfinite time interval  $(t_0, +\infty)$ . The matrix  $A = (a_{ij})$  of equation (2) is an obvious modification of the matrix  $\bar{A} = (\bar{a}_{ij})$  of equation (1), in which the constant elements  $\bar{a}_{ij}$  of  $\bar{A}$  are replaced by nonlinear, time-dependent functions  $a_{ij} = a_{ij}(t, x)$ .

To consider the connective aspect of stability, the elements  $a_{ij}$  of the matrix  $A$  in equation (1) are written:

$$a_{ij}(t, x) = -\delta_{ij}\psi_i(t, x) + e_{ij}(t) \psi_{ij}(t, x) \quad (3)$$

where  $\delta_{ij}$  is the Kronecker symbol and  $\psi_i(t, x), \psi_{ij}(t, x) \in C^{(0,0)}(T \times R^n)$ . In equation (3),  $e_{ij} = e_{ij}(t)$  are elements of the  $n \times n$  interconnection matrix  $E = (e_{ij})$ , which are  $e_{ij}(t) \in C^0(T)$  and are restricted as  $e_{ij}(t) \in [0, 1], \forall t \in T$  (ref. 12). For system (2), the element  $e_{ij}(t)$  reflects the coupling between  $x_i(t)$  and  $x_j(t)$  at each instant in time, that is, the time-dependent influence of the state  $x_j(t)$  on the derivative of the state  $x_i(t)$ . Therefore, the interconnection matrix  $E$  represents the structural perturbations of the nonlinear matrix system (2).

In this section, we study asymptotic stability properties of system (2) under structural perturbations. More precisely, we investigate stability formulated as:

*Definition 1. The equilibrium state  $x = 0$  of system (2) is connectively asymptotically stable in the large if and only if it is asymptotically stable in the large for all interconnection matrices  $E(t)$ .*

---

\*With some obvious exceptions, lower case roman letters denote vectors, capital Roman letters denote matrices, and Greek letters denote scalars.

Before we turn to the derivation of the conditions for the kind of stability expressed by definition 1, we need the notion of the *fundamental interconnection matrix*  $\bar{E}$  (ref. 12). The matrix  $\bar{E}$  is a time-invariant, interconnection matrix in which the elements  $\bar{e}_{ij}$  take on binary values  $-1$  if the  $j$ th state  $x_j$  influences the  $i$ th time derivative  $\dot{x}_i$  of the state  $x_i$  and  $0$  if  $x_j$  has no influence on  $\dot{x}_i$ . The matrix  $\bar{E}$  is a binary matrix (ref. 16) that reflects the basic structure of the system. Therefore, any interconnection matrix  $E(t)$  is generated from  $\bar{E}$  by replacing the unit elements of  $\bar{E}$  by corresponding elements  $e_{ij}(t)$  of  $E(t)$ .

The conditions for connective stability are expressed in terms of  $\bar{E}$ , but are valid for all  $E$  as required by definition 1. This is an important qualitative result since we show stability of a class of nonlinear, time-varying systems by proving stability of one member of that class which is a time-constant linear system.

To establish conditions for asymptotic connective stability, we assume that the elements  $a_{ij}(t, x)$  of the matrix  $A(t, x)$  are specified by equation (3) where  $\psi_i(t, x)$  and  $\psi_{ij}(t, x)$  are bounded functions on  $T \times R^n$  and that there exist numbers  $\alpha_i > 0$ ,  $\alpha_{ij} \geq 0$  such that

$$\psi_i(t, x)|x_i| > \alpha_i \phi_i(|x_i|), \quad \psi_{ij}(t, x)x_j \leq \alpha_{ij} \phi_j(|x_j|), \quad \forall i, j = 1, 2, \dots, n; \quad \forall (t, x) \in T \times R^n \quad (4)$$

and  $\alpha_i > \alpha_{ii}$ . In equation (4),  $\phi_i: R_+^1 \rightarrow R_+^1$  are comparison functions that belong to the class  $H: \phi_i(\rho) \in C^0(R_+^1)$ ,  $\phi_i(0) = 0$  and  $\phi_i(\rho) < \phi_i(\rho_2)$ ,  $\forall \rho_1, \rho_2: 0 \leq \rho_1 < \rho_2 < +\infty$  (ref. 17, 18).

By  $\bar{A} = (\bar{a}_{ij})$ , we denote the  $n \times n$  constant matrix with the coefficients

$$\bar{a}_{ij} = -\delta_{ij}\alpha_{ij} + \bar{e}_{ij}\alpha_{ij} \quad (5)$$

where the elements  $\bar{e}_{ij}$  take the values  $1$  or  $0$  according to the matrix  $\bar{E}$ .

We prove the following:

*Theorem 1. The equilibrium state  $x = 0$  of the system (2) is connectively asymptotically stable in the large if the  $n \times n$  constant matrix  $\bar{A} = (\bar{a}_{ij})$  defined by equations (4) and (5) satisfies the conditions:*

$$(-1)^k \begin{vmatrix} \bar{a}_{11} & \bar{a}_{12} & \dots & \bar{a}_{1k} \\ \bar{a}_{21} & \bar{a}_{22} & \dots & \bar{a}_{2k} \\ \dots & \dots & \dots & \dots \\ \bar{a}_{k1} & \bar{a}_{k2} & \dots & \bar{a}_{kk} \end{vmatrix} > 0, \quad \forall k = 1, 2, \dots, n. \quad (6)$$

*Proof:* Let us consider the function  $v: R^n \rightarrow R_+^1$ ,

$$v(x) = \sum_{i=1}^n b_i |x_i| \quad (7)$$

as a candidate for Liapunov's function (ref. 19) for system (2) where  $b_i > 0, i = 1, 2, \dots, n$  are yet unspecified numbers.

For  $v(x)$ , we have the inequalities

$$\phi_I(\|x\|) \leq v(x) \leq \phi_{II}(\|x\|), \quad \forall (t, x) \in T \times R^n \quad (8)$$

where  $\phi_I$  and  $\phi_{II} \in H$  are given as

$$\phi_I(\|x\|) = b_m \|x\|, \quad \phi_{II}(\|x\|) = n^{1/2} b_M \|x\| \quad (9)$$

and  $b_m = \min_i b_i$  and  $b_M = \max_i b_i$ .

Since the derivative of  $|x_i(t)|$  need not exist at a point where  $x_i(t) = 0$ , it is necessary to calculate the right-hand derivative  $D^+|x_i(t)|$  with respect to equation (2) as proposed in reference 19. For this purpose, the functional  $\sigma_i$  is defined as

$$\sigma_i = \left\{ \begin{array}{l} 1, \text{ if } x_i > 0, \text{ or if } x_i = 0 \text{ and } \dot{x}_i > 0 \\ 0, \text{ if } x_i = 0 \text{ and } \dot{x}_i = 0 \\ -1, \text{ if } x_i < 0, \text{ or if } x_i = 0 \text{ and } \dot{x}_i < 0, \end{array} \right\} \quad (10)$$

where  $x_i = x_i(t) \in C^1(T)$ . Then

$$D^+|x_i(t)| = \sigma_i \dot{x}_i(t). \quad (11)$$

Using the constraints (4) and expression (11), we calculate the desired derivative as

$$\left. \begin{aligned} D^+v(x) &= \sum_{i=1}^n b_i \sigma_i \dot{x}_i(t) \\ &= \sum_{i=1}^n b_i \sigma_i \sum_{j=1}^n a_{ij}(t, x) x_j(t) \\ &\leq b^T \bar{A} w(x), \quad \forall (t, x) \in T \times R^n \end{aligned} \right\} \quad (12)$$

where  $b = (b_1 \ b_2 \ \dots \ b_n)^T$  is a positive constant vector ( $b > 0$ ) and the positive vector function  $w: R^n \rightarrow R_+^n$  is defined as

$$w(x) = [\phi_1(|x_1|) \ \phi_2(|x_2|) \ \dots \ \phi_n(|x_n|)]^T. \quad (13)$$

From equation (5), we conclude that  $\bar{A}$  is a Metzler matrix (ref. 20), that is, it has negative diagonal elements ( $\bar{a}_{ii} < 0$ ) and nonnegative off-diagonal elements ( $\bar{a}_{ij} \geq 0, i \neq j$ ). We recall that a matrix  $\bar{A}$  is called a Hicks matrix (ref. 3) (equivalently,  $-\bar{A}$  is an  $M$  matrix (ref. 21) if all even-order

principle minors of  $\bar{A}$  are positive and all odd-order principle minors of  $\bar{A}$  are negative. For a Metzler matrix  $\bar{A}$ , the Hicksian property is equivalent to the Sevastyanov-Kotelyanskii conditions (6) (ref. 22). Since  $\bar{A}$  is a Metzler matrix, the fact that  $\bar{A}$  satisfies inequalities (6) and is a Hicks matrix is equivalent (refs. 20, 21) to saying that, for any constant vector  $c > 0$ , there exists a constant vector  $b > 0$  such that

$$c^T = -b^T \bar{A}. \quad (14)$$

Therefore, we can rewrite inequality (12) as

$$\begin{aligned} D^+v(x) &\leq -c^T w(x) \\ &\leq -c_m \sum_{i=1}^n \phi_i(|x_i|) \\ &\leq -\phi_{III}(\|x\|), \quad \forall(t, x) \in T \times R^n \end{aligned} \quad (15)$$

where  $c_m = \min_i c_i$  and  $\phi_{III}(\|x\|) \in H$ .

From (8) and (15) and reference 18, we conclude global asymptotic stability of  $x = 0$  in equation (2). To show that stability is also connective, we need only notice that

$$A(t, x)x \leq \bar{A}w(x), \quad \forall(t, x) \in T \times R^n \quad (16)$$

where inequality (16) is taken component-wise. Therefore, equation (15) holds for all  $E(t)$ . This proves theorem 1.

Note that the constraints (4) imply that  $\psi_i(t, x) > 0$ ,  $\forall(t, x) \in T \times R^n$ . Positivity of  $\psi_i(t, x)$  is absolutely essential for stability of equation (1) since it is easy to show that the Hicks conditions (6) imply  $\bar{a}_{ij} < 0$ ,  $\forall i = 1, 2, \dots, n$ . With this in mind, we can rewrite the first condition in (4) as  $|\psi_i(t, x)x_j| \geq \alpha_i \phi_{ij}(|x_j|)$ , which looks similar to the second condition (4) except for the reversal of the inequality sign.

If the conditions (4) are simplified to

$$\psi_i(t, x) > \alpha_i, \quad |\psi_{ij}(t, x)x_j| \leq \alpha_{ij}|x_j| \quad \forall i, j = 1, 2, \dots, n; \quad \forall(t, x) \in T \times R^n \quad (17)$$

where comparison functions  $\phi_j(|x_j|)$  are chosen as  $|x_j|$ , then we can establish exponential property of connective stability as in reference 9. Furthermore, if we use the notion of absolute stability for the nonlinear matrix systems proposed by Persidskii (ref. 23), we can prove that Sevastyanov-Kotelyanskii inequalities (6) become both necessary and sufficient conditions for connective stability (ref. 9).

On the basis of the constraints (17), we define the following classes of continuous functions:

$$\left. \begin{aligned} \psi_i &= \{\psi_i(t, x) : \psi_i(t, x) \geq \alpha_i\} \\ \psi_{ij} &= \{\psi_{ij}(t, x) : |\psi_{ij}(t, x)| \leq \alpha_{ij}\} \end{aligned} \right\} \quad (18)$$

where  $\alpha_i, \alpha_{ij}$  are numbers as in (4). Then, we state:

*Definition 2.* The equilibrium state  $x = 0$  of the system equation (2) is connectively, absolutely, and exponentially stable if and only if there exist two positive numbers  $\Pi$  and  $\pi$  independent of initial conditions  $(t_0, x_0)$  such that

$$\|x(t; t_0, x_0)\| \leq \Pi \|x_0\| \exp[-\pi(t - t_0)], \quad \forall t \in T_0 \quad (19)$$

for all  $(t_0, x_0) \in T \times R^n$ , all  $\psi_i \in \Psi_i$ ,  $\psi_{ij} \in \Psi_{ij}$ , and all interconnection matrices  $E(t)$ .

To establish this kind of stability, we can use the following:

*Theorem 2.* The equilibrium state  $x = 0$  of the system (2) is connectively, absolutely, and exponentially stable if and only if the  $n \times n$  constant matrix  $\bar{A} = (\bar{a}_{ij})$  defined by (5) and (17) satisfies conditions (6).

*Proof:* Let us consider again the function  $v(x)$  in equation (7). When (4) is reduced to (17), the vector  $w(x)$  in equation (13) becomes  $[|x_1| |x_2| \dots |x_n|]^T$ , and from (12) and (16) we get

$$D^+v(x) \leq \sum_{j=1}^n b_j |x_j| |\bar{a}_{jj}| + \sum_{j=1}^n |x_j| \sum_{\substack{i=1 \\ i \neq j}}^n b_i |\bar{a}_{ij}|, \quad \forall (t, x) \in T \times R^n. \quad (20)$$

Since  $\bar{A}$  is a Metzler matrix, the fact that it satisfies conditions (6) is equivalent (refs. 21, 24) to saying that there exists a positive vector  $b = (b_1 \ b_2 \ \dots \ b_n)^T$ , and a positive number  $\pi$  such that

$$|\bar{a}_{jj}| - b_j^{-1} \sum_{\substack{i=1 \\ i \neq j}}^n b_i |\bar{a}_{ij}| \geq \pi, \quad \forall j = 1, 2, \dots, n \quad (21)$$

that is,  $\bar{A}$  is a quasidominant diagonal matrix (ref. 24).

From (20) and (21), we get the differential inequality

$$D^+v \leq -\pi v, \quad \forall t \in T_0, \quad \forall v \in R_+^1, \quad \forall E. \quad (22)$$

By integrating equation (22), we obtain

$$v[x(t)] \leq v(x_0) \exp[-\pi(t - t_0)], \quad \forall t \in T_0, \quad \forall (t_0, x_0) \in T \times R^n, \quad \forall E. \quad (23)$$

Using the well-known relationship between the Euclidean and absolute-value norms  $\|x\| \leq |x| \leq n^{1/2} \|x\|$ , we can rewrite (23) as

$$\left. \begin{aligned} \|x(t; t_0, x_0)\| &\leq \Pi \|x_0\| \exp[-\pi(t - t_0)] , \quad \forall t \in T_0 \\ V(t_0, x_0) &\in T \times R^n , \quad \forall \psi_i \in \Psi_i , \quad \forall \psi_{ij} \in \Psi_{ij} , \quad \forall E \end{aligned} \right\} \quad (24)$$

with

$$\Pi = n^{1/2} b_M b_m^{-1} \quad (25)$$

where  $b_M = \max_i b_i$  and  $b_m = \min_i b_i$ .

Therefore, conditions (6) are sufficient for the absolute exponential property of connective stability of the equilibrium  $x = 0$  in equation (2). This establishes the "if" part of theorem 2.

To prove the "only if" part of theorem 2, we select the particular system (2) specified by

$$\psi_i(t, x) = \alpha_i , \quad \psi_{ij}(t, x) = \alpha_{ij} , \quad \forall i, j = 1, 2, \dots, n \quad (26)$$

and the fundamental interconnection matrix  $\tilde{E}$ . That is, the matrix  $A(t, x)$  in equation (2) is taken as the constant Metzler matrix  $\tilde{A}$  and system (2) is described by equation (1). If  $\tilde{A}$  does not satisfy the Hicks conditions (6), the system (1) is unstable, and the equilibrium  $x = 0$  of equation (2) is not stable  $\forall \psi_i \in \Psi_i , \forall \psi_{ij} \in \Psi_{ij}$ . This completes the proof of theorem 2.

## HIERARCHIC MODELS

On the basis of the results obtained in the preceding section, we conclude that stability is a highly reliable property of competitive dynamic systems since it can tolerate a wide range of structural, nonlinear, and time-varying perturbations. Therefore, it would be desirable to define a class of noncompetitive dynamic systems for which a competitive model can be constructed so that their structural stability properties are implied by the same properties of the model. Such a construction is possible in the context of the hierarchic stability analysis, and we will show how the decomposition-aggregation scheme (refs. 11–14) can be used to form a competitive aggregate model for a large class of dynamic systems. The scheme is based on the modern mathematical machinery of the comparison principle and vector Liapunov functions (ref. 15). The decomposition-aggregation stability analysis not only provides a competitive aggregate model with structural stability properties, but it can also take advantage of the special structural features of complex dynamic systems and yield considerable simplification in the relevant stability investigations.

Let us immediately recall the fact stated in reference 7 that a natural generalization of the competitive models considered in reference 3 is represented by a differential equation

$$\dot{z} = u(t, z) \quad (27)$$

where the function  $u : T \times R^s \rightarrow R^s$  is  $u(t, x) \in C^{(0,0)}(T \times R^n)$  and  $u(t, 0) = 0, \forall t \in T$ . The function  $u(t, x)$  belongs to the following class of functions:

$$K: u_i(t, a) \leq u_i(t, b), \quad \forall i = 1, 2, \dots, s$$

$$\forall \{(t, a), (t, b)\} \in T \times R^s \ni a_i = b_i, a_j \leq b_j (j = 1, 2, \dots, s; i \neq j).$$

The class of functions  $K$ , used by Kamke (ref. 25) in a formulation of the comparison principle, plays an important role in the stability analysis of dynamic systems by the vector Liapunov function. Therefore, the strong stability results obtained in the theory of differential inequalities and vector Liapunov functions can now be used to study stability properties of the competitive models.

In the following development, we will show how a competitive model described by equation (27) can be constructed for a dynamic system,

$$\dot{x} = f(t, x). \quad (28)$$

In (28),  $x(t) \in R^n$  is the state vector of the system and the function  $f: T \times R^n \rightarrow R^n$  is defined, bounded, and continuous on the domain  $T \times R^n$  so that the solution  $x(t; t_0, x_0)$  of equation (28) exists for all initial conditions  $(t_0, x_0) \in T \times R^n$  and  $t \in T_0$ . Furthermore, we assume that  $f(t, 0) = 0, \forall t \in T$ , and  $x = 0$  is the unique equilibrium state of the system (28). We derive conditions under which stability of the trivial solution  $z = 0$  of equation (27) implies connective stability of the equilibrium  $x = 0$  of the system (28).

To introduce the connective aspect of stability in the context of system (28), let us represent the state space  $R^n$  as

$$R^n = R_1^{n_1} \times R_2^{n_2} \times \dots \times R_s^{n_s} \quad (29)$$

so that the state vector has the form

$$x(t) = [x_1^T(t) x_2^T(t) \dots x_s^T(t)]^T \quad (30)$$

and  $x_i(t) \in R^{n_i}$ . The function  $f(t, x)$  is further specialized by its components as

$$f_i(t, x) \equiv \hat{f}_i(t, x_i, e_{i1}x_1, e_{i2}x_2, \dots, e_{is}x_s), \quad i = 1, 2, \dots, s \quad (31)$$

where  $e_{ij}(t)$  are elements of the  $s \times s$  interconnection matrix  $E(t)$ . Now, we can broaden the scope of definition 1 to include the class of systems (28). We say that the equilibrium  $x = 0$  of the system (28) is connectively stable if and only if it is stable (in the sense of Liapunov) for all interconnection matrices  $E(t)$  (ref. 12).

To establish connective stability of the system, we will form the model (eq. (27)) where the "aggregate" function  $u(t, x)$  has the form

$$u_i(t, z) \equiv \hat{u}_i(t, z_i, e_{i1}z_1, e_{i2}z_2, \dots, e_{is}z_s), \quad i = 1, 2, \dots, s \quad (32)$$

and  $z_i \in R^1$ . On the basis of (32), we define the model



$$\dot{z} = \bar{u}(t, z) \quad (33)$$

where the function  $\bar{u}(t, z) \in K$  corresponds to the fundamental interconnection matrix  $\bar{E}$ . Then, stability of the trivial solution  $z = 0$  of the aggregate competitive model (33) implies connective stability of the equilibrium  $x = 0$  of the system equation (28).

More precisely, we prove the following:

*Theorem 3. There exists a function  $v(t, x)$  with the properties:  $v(t, x) \in C^{(0,0)}(T \times R^n)$ ;  $v(t, x)$  is locally Lipschitzian in  $x$ ;  $v(t, 0) \equiv 0$ , and  $v(t, x) \geq 0$  on  $T \times R^n$ ; the function  $v: T \times R^n \rightarrow R_+^1$ , defined as*

$$v(t, x) = b^T v(t, x) = \sum_{i=1}^s b_i v_i(t, x) \quad (34)$$

for some constant vector  $b > 0$  satisfies the inequalities

$$\phi_I(\|x\|) \leq v(t, x) \leq \phi_{II}(\|x\|), \quad V(t, x) \in T \times R^n \quad (35)$$

where  $\phi_I, \phi_{II} \in H$  and  $\phi_I(\rho) \rightarrow +\infty$  as  $\rho \rightarrow +\infty$ ; the function

$$D^+ v(t, x) = \limsup_{h \rightarrow 0^+} \frac{1}{h} \{v[t+h, x+hf(t, x)] - v(t, x)\} \quad (36)$$

defined with respect to equation (28), satisfies a differential inequality

$$D^+ v(t, x) \leq \bar{u}[t, v(t, x)], \quad V(t, v) \in T \times R^s \quad (37)$$

where the function  $\bar{u}: T \times R_+^s \rightarrow R^s$  is  $\bar{u}(t, v) \in K$  and  $\bar{u}(t, 0) = 0$ .

Then, asymptotic stability in the large of  $z = 0$  in the comparison equation (33) implies connective asymptotic stability in the large of  $x = 0$  in equation (28), and

$$v(t, x) = [v_1(t, x) \ v_2(t, x) \ \dots \ v_s(t, x)]^T \quad (38)$$

is a vector Liapunov function for the system (28).

*Proof:* It is a well-known result (ref. 15) that under the conditions of the theorem, asymptotic stability of  $z = 0$  of equation (33) implies the same stability property for  $x = 0$  in equation (28). To show that stability of  $x = 0$  is also connective, we notice that the assumption  $v(t, x) \geq 0$  on  $T \times R^n$  allows us to establish the inequality

$$u(t, z) \leq \bar{u}(t, z), \quad V(t, z) \in T \times R^s \quad (39)$$

which holds for all  $E(t)$ . Therefore, the differential inequality in equations (37) is also valid for all interconnection matrices  $E(t)$ , and stability of  $x = 0$  is connective. This proves theorem 3.

An attractive property of theorem 3 is that it allows a reduction in dimensionality of stability problems in much the same way as does the original Liapunov theory. By theorem 3, we can prove stability of a system of order  $n$  by demonstrating stability of  $s$  lower-order systems and one system of order  $s \leq n$  which involves the vector Liapunov function.

Another important feature of theorem 3 we will not take advantage of is that the components  $v_i(t, x)$  of the vector Liapunov function  $v(t, x)$  can satisfy the weaker requirements than the usual ones associated with scalar Liapunov functions. This is not to say, however, that vector Liapunov functions are automatically more flexible than scalar Liapunov functions as was demonstrated in reference 7.

Let us now show how to construct the competitive aggregate model (27) for a given system (28) by applying the decomposition-aggregation method proposed in reference 13. We start with decomposing the system (28) into  $s$  subsystems described by

$$\dot{x}_i = g_i(t, x_i) + h_i(t, x), \quad i = 1, 2, \dots, s \quad (40)$$

where  $x_i(t) \in R^{n_i}$  is the state vector of the subsystem (40) and represents the  $i$ th "vector" component of the state vector  $x(t)$  specified by equation (30).

In equation (40), the functions  $g_i : T \times R^{n_i} \rightarrow R^{n_i}$  describe the "decoupled" subsystems:

$$\dot{x}_i = g_i(t, x_i), \quad i = 1, 2, \dots, s \quad (41)$$

obtained from equation (40) when all interaction functions  $h_i : T \times R^n \rightarrow R^{n_i}$  among the subsystems are set to zero ( $h_i(t, x) \equiv 0$ ). Each  $g_i(t, 0) = 0, \forall t \in T$ , so that  $x_i = 0$  is the unique equilibrium state of every subsystem (41).

We assume that with each *decoupled subsystem* (41) we can associate a function  $v_i : T \times R^{n_i} \rightarrow R_+^1$  such that  $v_i(t, x_i) \in C^{(0,0)}(T \times R^{n_i})$ ,  $v_i(t, x)$  satisfies a Lipschitz condition in  $x_i$  for a constant  $\kappa_i > 0$ , and

$$\left. \begin{aligned} \phi_{i1}(\|x_i\|) &\leq v_i(t, x_i) \leq \phi_{i2}(\|x_i\|) \\ D^+ v_i(t, x_i)_{(41)} &\leq -\phi_{i3}(\|x_i\|) \\ \forall i = 1, 2, \dots, s; \quad \forall t \in T, \quad \forall x_i \in R^{n_i} \end{aligned} \right\} \quad (42)$$

where  $D^+ v_i(t, x_i)_{(41)} = \limsup_{h \rightarrow 0^+} \frac{1}{h} \{v_i[t+h, x_i + hg_i(t, x_i)] - v_i(t, x_i)\}$ , the functions  $\phi_{i1}, \phi_{i2}, \phi_{i3} \in H$  and  $\phi_{i1}(\rho) \rightarrow +\infty$  as  $\rho \rightarrow +\infty$ .

We assume that the *interactions*  $h_i(t, x)$  among the subsystems (41) have the form

$$h_i(t, x) \equiv h_i(t, e_{i1}x_1, e_{i2}x_2, \dots, e_{is}x_s) \quad (43)$$

and that there exist bounded functions  $\xi : T \times R^n \rightarrow R_+^1$  such that

$$\left. \begin{aligned} \|h_i(t, x)\| &\leq \sum_{j=1}^s e_{ij}(t) \xi(t, x) \phi_{j3}(\|x_j\|), \\ \forall i = 1, 2, \dots, s; \quad \forall (t, x) &\in T \times R^n \end{aligned} \right\} \quad (44)$$

Let us define an  $s \times s$  constant matrix  $\bar{A} = (\bar{a}_{ij})$  by

$$\bar{a}_{ij} = -\delta_{ij} + \bar{e}_{ij} \kappa_i \alpha_{ij} \quad (45)$$

where  $\bar{e}_{ij}$  are the elements of the fundamental interconnection matrix  $\bar{E}$ , and the numbers  $\alpha_{ij} \geq 0$  are computed as

$$\alpha_{ij} = \max \left\{ 0, \sup_{T \times R^n} \xi_{ij}(t, x) \right\} \quad (46)$$

Now we state:

*Theorem 4. The equilibrium state  $x = 0$  of the system (28) is connectively asymptotically stable in the large if the  $s \times s$  constant matrix  $\bar{A} = (\bar{a}_{ij})$  defined by equation (45) satisfies the conditions (6).*

*Proof:* As in reference 13, consider the function  $v : T \times R^n \rightarrow R_+^1$ ,

$$v(t, x) = \sum_{i=1}^s b_i v_i(t, x_i) \quad (47)$$

as a candidate for a Liapunov function for the system (28) where  $b_i > 0, i = 1, 2, \dots, s$  are components of a positive (yet unspecified) vector  $b$ . Then define the function

$$D^+ v_i(t, x_i)_{(40)} = \limsup_{h \rightarrow 0^+} \frac{1}{h} \{v_i(t+h, x_i+h[g_i(t, x_i)+h_i(t, x)]) - v_i(t, x_i)\} \quad (48)$$

which we compute with respect to subsystem equation (40) and obtain

$$D^+ v_i(t, x_i)_{(40)} \leq D^+ v_i(t, x_i)_{(41)} + \kappa_i \|h_i(t, x)\|, \quad i = 1, 2, \dots, s \quad (49)$$

since each  $v_i(t, x_i)$  is Lipschitzian with a constant  $\kappa_i > 0$  (ref. 15).

By using the interconnection conditions (44) and subsystem stability equations (42), from (49) we obtain the vector differential inequality:

$$D^+ v_{(40)} \leq \bar{A} w(v), \quad \forall (t, x) \in T \times R^n \quad (50)$$

which holds for all interconnection matrices  $E(t)$ .

As in equation (13),  $w : R^S \rightarrow R^S$  is the comparison vector function:

$$w(v) = [\phi_{13}(v_1) \phi_{23}(v_2) \dots \phi_{s3}(v_s)]^T. \quad (51)$$

By using the same argument as in the proof of theorem 1, from equations (47) and (50), we obtain

$$\phi_I(\|x\|) \leq v(t, x) \leq \phi_{II}(\|x\|), \quad D^+v(t, x) \leq -\phi_{III}(\|x\|), \quad V(t, x) \in T \times R^n \quad (52)$$

where the functions

$$\phi_I(\|x\|) = b_m \sum_{i=1}^s \phi_{i1}(\|x_i\|),$$

$$\phi_{II}(\|x\|) = b_M \sum_{i=1}^s \phi_{i2}(\|x_i\|),$$

$$\phi_{III}(\|x\|) = c_m \sum_{i=1}^s \phi_{i3}(\|x_i\|)$$

all belong to the class  $H$ .

Since inequalities (52) hold for all interconnection matrices  $E(t)$ , the equilibrium state  $x = 0$  of the system (28) is connectively asymptotically stable in the large. The proof of theorem 4 is complete.

By observing that  $\bar{A}w(v) \in K$ , theorem 4 follows directly from inequality (50) and theorem 3. In the proof of theorem 4, we used the scalar function  $v = b^T v$  to show explicitly the multilevel nature of the analysis. The components  $v_i$  of the vector Liapunov function  $v$  are scalar Liapunov functions responsible for stability on the subsystem level, and the scalar function  $v$  is a scalar Liapunov on the aggregate overall system level.

If we strengthen the constraints imposed on the interactions  $h_i(t, x)$  among the subsystems, we can infer exponential stability of the overall system from exponential stability of the subsystems.

Let us assume that the estimates (42) have the form

$$\left. \begin{aligned} \eta_{i1} \|x_i\| \leq v_i(t, x_i) \leq \eta_{i2} \|x_i\| \quad D^+v_i(t, x_i) \leq -\pi_i v_i(t, x_i), \\ \forall i = 1, 2, \dots, s; \quad \forall t \in T, \quad \forall x_i \in R^{n_i} \end{aligned} \right\} \quad (53)$$

where  $\eta_{i1}$ ,  $\eta_{i2}$ , and  $\pi_i$  are all positive numbers. Inequalities (53) guarantee exponential stability of the decoupled subsystems (39).

We also assume that the interactions  $h_i(t, x)$  among the subsystems satisfy the constraints

$$\|h_i(t, x)\| \leq \sum_{j=1}^s e_{ij}(t) \xi_{ij}(t, x) \|x_j\| \quad \forall i = 1, 2, \dots, s; \quad \forall (t, x) \in T \times R^n. \quad (54)$$

We again construct the aggregate  $s \times s$  constant matrix  $\bar{A} = (\bar{a}_{ij})$  defined by

$$\bar{a}_{ij} = -\delta_{ij}\pi_i + \bar{e}_{ij}\alpha_{ij}\kappa_i\eta_{j1}^{-1} \quad (55)$$

and prove the following:

*Theorem 5. The equilibrium state  $x = 0$  of the system (28) is connectively exponentially stable in the large if the  $s \times s$  constant matrix  $\bar{A} = (\bar{a}_{ij})$  defined by equation (55) satisfies the inequalities (6).*

*Proof:* We again use the function  $v(t, x)$  of equation (47) as a Liapunov function. By using the estimates (53) and (54), from (49) we obtain the following inequalities:

$$\left. \begin{aligned} D^+v_i(t, x_i)_{(40)} &\leq -\pi_i v_i(t, x_i) + \kappa_i \sum_{j=1}^s \bar{e}_{ij}\alpha_{ij}\|x_j\| \\ &\leq -\pi_i v_i(t, x_i) + \kappa_i \sum_{j=1}^s \bar{e}_{ij}\alpha_{ij}\eta_{j1}^{-1} v_j(t, x_j) \\ i &= 1, 2, \dots, s; \quad \forall t \in T, \quad \forall x \in R^n \end{aligned} \right\} \quad (56)$$

that are valid for all interconnection matrices  $E(t)$ .

From equation (56), we obtain the aggregate competitive model:

$$D^+v_{(40)} \leq \bar{A}v, \quad \forall (t, x) \in T \times R^n \quad (57)$$

which is a linear differential inequality in  $v$ . Proceeding as in the proof of theorem 4, we derive the scalar inequality

$$D^+v(t, x)_{(40)} \leq -\pi v(t, x), \quad \forall (t, x) \in T \times R^n, \quad \forall E(t) \quad (58)$$

where  $\pi = \min_i \pi_i$ .

Integrating inequality (58), we obtain

$$v[t, x(t)] \leq v(t_0, x_0) \exp[-\pi(t - t_0)], \quad \forall t \in T_0, \quad \forall (t_0, x_0) \in T \times R^n, \quad \forall E(t). \quad (59)$$

By use of the estimates (53), we can obtain from this inequality a further inequality that involves the solution  $x(t; t_0, x_0)$  of the original overall system:

$$\|x(t; t_0, x_0)\| \leq \Pi \|x_0\| \exp[-\pi(t - t_0)] , \quad \forall t \in T_0 , \quad V(t_0, x_0) \in T \times R^n , \quad \forall E(t) \quad (60)$$

where  $\Pi = s^{1/2} b_m b_M \eta_{m1}^{-1} \eta_{M2}$  ,  $b_m = \min_i b_i$  ,  $b_M = \max_i b_i$  ,  $\eta_{m1} = \min_i \eta_{i1}$  ,  $\eta_{M2} = \max_i \eta_{i2}$  , and  $\pi = \min_i \pi_i$  . This proves theorem 5.

## CONCLUSIONS

Two important conclusions result from this work. First, the competitive structures are an appropriate framework for constructing reliable complex systems. Secondly, via the diagonal dominance of the aggregate competitive model, the decomposition-aggregation method provides a good measure of complexity for stable, large-scale systems. Both results arose in the context of connective stability by applying the mathematical apparatus of the comparison principle and vector Liapunov function. One of the most attractive aspects of the results is that they are obtained in the Hicks-Metzler algebraic setting, which provides a rich environment for their application and further refinements.

A number of interesting problems were either not mentioned or were not explored in sufficient detail. For example, it is not clear how the decomposition and aggregation should be performed to balance the gains in simplification against the errors resulting from the approximation involved in the decomposition-aggregation process. Furthermore, it is of interest to investigate various implications (ref. 26) of the obtained results in the competitive analysis conducted in the field of economics (refs. 2, 3, 7) and ecosystems (refs. 4, 8, 10). In control systems (ref. 27), there are already some specific applications that indicate certain definite advantages of the competitive analysis and the decomposition-aggregation method in the multilevel stabilization and optimization of large-scale systems. A good deal of work remains to be done to explore the possibilities offered by the competitive structures in dynamic systems and to obtain important new results.

## REFERENCES

1. Simon, H. H.: The Architecture of Complexity, *Proceedings of the American Philosophical Society*, 106(1962), 467–482.
2. Quirk, J.; and Saposnik, R.: *Introduction to General Equilibrium Theory and Welfare Economics*, McGraw-Hill, New York, 1968.
3. Arrow, K. J.; and Hahn, F. H.: *General Competitive Analysis*, Holden-Day, San Francisco, 1971.
4. May, R. M.: *Stability and Complexity in Model Ecosystems*, *Monographs in Population Biology*, No. 6, Princeton University Press, N. J., 1973.
5. Richardson, L. F.: *Arms and Insecurity*, Boxwood Press, Pittsburgh, 1960.
6. Bellman, R.: Topics in Pharmacokinetics, I: Concentration-Dependent Rates, *Mathematical Biosciences*, 6(1970), 13–17.
7. Šiljak, D. D.: Competitive Economic Systems: Stability, Decomposition, and Aggregation, *Proc. of the 1973 Conference on Decision and Control*, San Diego, 1973, 265–275.
8. Šiljak, D. D.: Connective Stability of Complex Ecosystems, *Nature*, 249(1974), 280.
9. Šiljak, D. D.: On Connective Stability and Instability of Competitive Equilibrium, *Proc. 1974 JACC Conference*, University of Texas, Austin, Texas, 1974, 223–228.
10. Šiljak, D. D.: When is a Complex Ecosystem Stable?, *Mathematical Biosciences*, (to be published).
11. Šiljak, D. D.: Stability of Large-Scale Systems Under Structural Perturbations, *IEEE Transactions*, SMC-2(1972), 657–663.
12. Šiljak, D. D.: On Stability of Large-Scale Systems Under Structural Perturbations, *IEEE Transactions*, SMC-3(1973), 415–417.
13. Grujić, Lj. T.; and Šiljak, D. D.: Asymptotic Stability and Instability of Large-Scale Systems, *IEEE Transactions*, AC-18(1973), 636–645.
14. Grujić, Lj. T.; and Šiljak, D. D.: Exponential Stability of Large-Scale Discrete Systems, *International Journal of Control*, 19(1974), 481–491.
15. Lakshmikantham, V.; and Leela, S.: *Differential and Integral Inequalities*, Vols. I and II, Academic Press, New York, 1969.
16. Harray, F.; Norman, R.; and Cartwright, D.: *Structural Models: An Introduction to the Theory of Directed Graphs*, Wiley, New York, 1965.
17. Hahn, W.: *Stability of Motion*, Springer-Verlag, New York, 1967.
18. Halanay, A.: *Differential Equations: Stability, Oscillations, Time Lags*, Academic Press, New York, 1966.

19. Rosenbrock, H. H.: A Lyapunov Function for Some Naturally Occurring Linear Homogeneous Time-Dependent Equations, *Automatica*, 1(1963), 97–109.
20. Arrow, K. J.: Price-Quality Adjustments in Multiple Markets with Rising Demands, *Proceedings of a Symposium on Mathematical Methods in the Social Sciences*, Stanford University Press, Stanford, California, 1966, pp. 3–15.
21. Fiedler, M.; and Pták, V.: On Matrices with Non-Positive Off-Diagonal Elements and Positive Principal Minors, *Czechoslovakian Mathematical Journal*, 12(1969), 382–400.
22. Gantmacher, F. R.: *The Theory of Matrices*, Vol. II, Chelsea, New York, 1960.
23. Persidskii, S. K.: Problem of Absolute Stability (in Russian), *Avtomatika i Telemekhanika*, 29(1969), 5–11.
24. Newman, P. K.: Some Notes on Stability Conditions, *Review of Economic Studies*, 72(1959), 1–9.
25. Kamke, E.: Zur Theorie der Systeme gewöhnlicher Differentialgleichungen, II, *Acta Mathematica*, 58(1932), 57–85.
26. Šiljak, D. D.: Connective Stability of Nonlinear Matrix Systems, *Proc. 1974 IEEE International Symposium on Circuits and Systems*, San Francisco, Calif., 1974, 37–41.
27. Čuk, S. M.; and Šiljak, D. D.: Decomposition-Aggregation Stability Analysis of the Spinning Skylab, *Proc. Seventh Asilomar Conference on Circuits, Systems, and Computers*, Pacific Grove, Calif., 1973, 204–211.